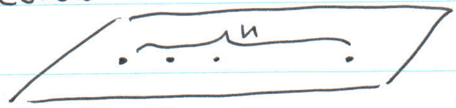
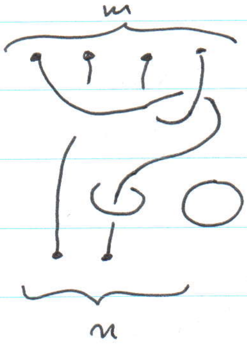
0511 Landa A homological invariant of tangles and tangle cobordisms

review



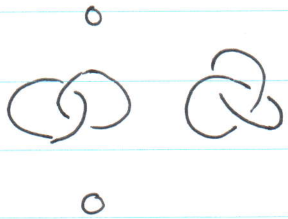
$\xmapsto{J} \quad V^{\otimes n}$



$\longmapsto$

$$\begin{array}{c} V^{\otimes m} \\ \uparrow J(T) \\ V^{\otimes n} \end{array}$$

$J(T)$ : intertwines the $U_q(sl_2)$-action

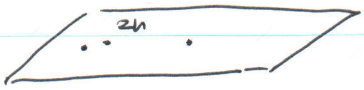


$\longmapsto$

$$\begin{array}{c} \mathbb{Z}[t,t^{-1}] \\ \uparrow J(L) \\ \mathbb{Z}[t,t^{-1}] \end{array}$$

Jones polynomial

A different version



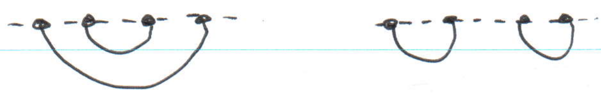
$\xmapsto{J'} \quad Inv(n) = Hom_{U_q(sl_2)}(\mathbb{C}, V^{\otimes 2n}) \subset V^{\otimes 2n}$

$Inv(n)$ has basis $B^n$ making it a free $\mathbb{Z}[q,q^{-1}]$-module

The basis $B^n$ consists of crossingless matches of $2n$ marked pts

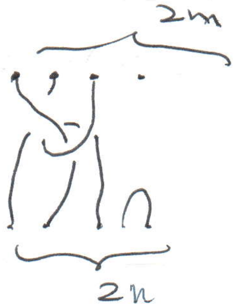
$B^0 = \phi$

$B^1 =$ 

$B^2 =$ 

$B^3 =$ 

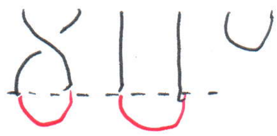
In general $|\mathbb{B}^n| = \frac{1}{n+1}\binom{2n}{n}$ ← ... the $n^{th}$ Catalan number

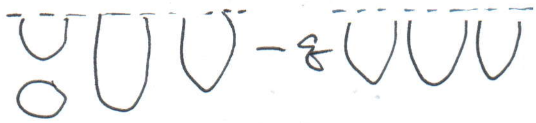



$\xrightarrow{J'}$

$$\begin{array}{c} Inv(m) \\ \uparrow \\ Inv(n) \end{array} \quad J'(T) = J(T)\big|_{Inv(n)}$$

__Example__

 $\xrightarrow{J}$ 

= 

= $(q+q^{-1})$  $- q$ 

= $q^{-1}$  $\in Inv(3)$

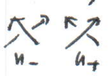
normalization

$\langle \phi \rangle = 1$

$\langle O \rangle = q + q^{-1}$

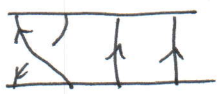
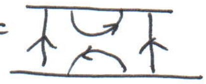
$\langle \times \rangle = \langle \asymp \rangle - q \langle )( \rangle$

$J'(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$




N.B. orientation

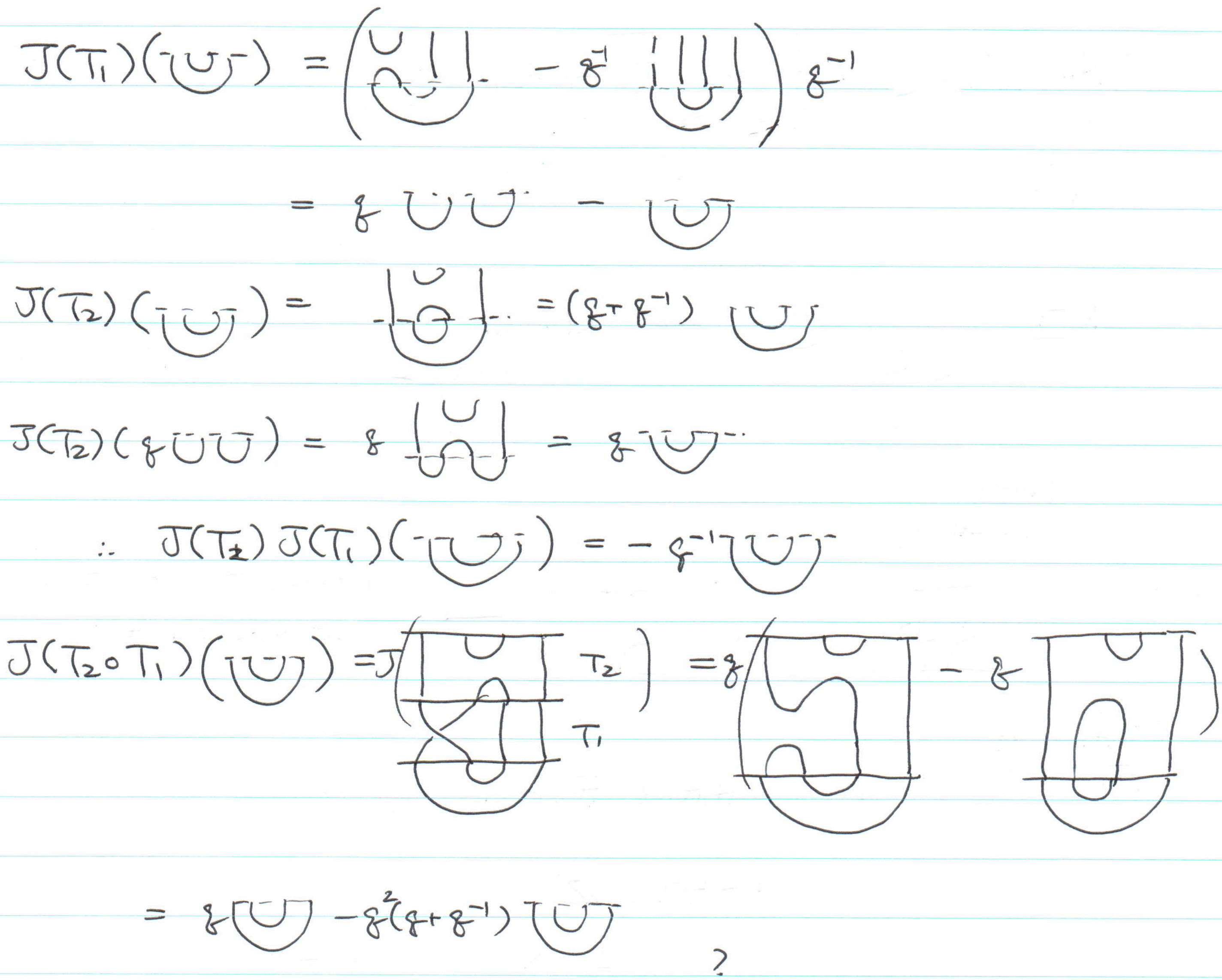
__Compatible with__
Composition ?

$T_1 =$  $T_2 =$ 

$n_+ = 1, n_- = 0$

check that $J(T_2) \circ J(T_1) = J(T_2 \circ T_1)$

Let's check this identity on  $\in B^2$

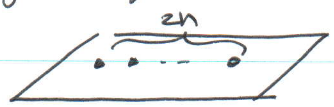



<u>Exercise</u> Find the mistake!

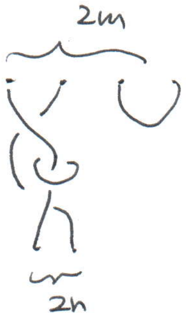
<u>Th</u>. The above construction is a functor

$$\mathrm{Tang} \longrightarrow \text{category of free } \mathbb{Z}[q, q^{-1}]\text{-modules and module maps}$$

By categorification I mean




$\longmapsto$ category $K_n$ complexes of $H^n$-modules for some ring $H^n$

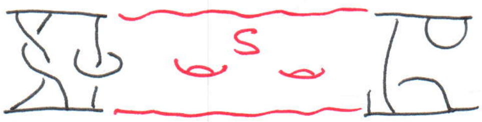


functor $\mathcal{H}_n \to \mathcal{H}_m$

<u>Note</u>

$\longmapsto$ Khovanov homology $\qquad H^0 = \mathbb{Z}[q, q^{-1}]$

tangle cobordism

$\longmapsto$ natural transformation between functors

<u>Definition of $H^n$</u>

a commutative Frobenius ring $A$

$$A = H^2(S^2; \mathbb{Z})$$

basis $1, X \qquad X^2 = 0$

trace $\quad \varepsilon(X) = 1, \quad \varepsilon(1) = 0$

comm. Frob. $\Longleftrightarrow$ trace is non-degenerate

$\forall a \neq 0 \ \exists b \quad \varepsilon(ab) \neq 0$

This makes $A \cong A^*$ as $A$-module

$m : A \otimes A \longrightarrow A$ $\quad$ multiplication

$\Delta : A \longrightarrow A \otimes A$ $\quad$ comult.

$\Delta(1) = 1 \otimes X + X \otimes 1$

$\Delta(X) = X \otimes X$

comm. Frobenius ring $\longleftrightarrow$ (1+1)-dimensional topological quantum field theory $\quad 2\mathrm{Cob} \xrightarrow{T}$ Vector sp or abelian groups

2Cob. : category of oriented 2-dim. cobordisms between closed oriented 1-manifolds

$\xrightarrow[\text{tensor functor}]{\mathcal{T}}$ Vect or Ab

* Diffeomorphic 2-dim. cobordisms map to the same linearmaps / homomorphisms

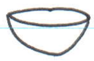
On objects

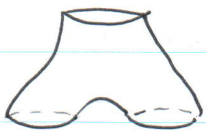
$\bigcirc \overset{\mathcal{T}}{\longmapsto} A$ $\qquad \underbrace{\bigcirc\ \bigcirc\ \bigcirc}_{k} \overset{\mathcal{T}}{\longmapsto} \underbrace{A \otimes A \otimes \cdots \otimes A}_{k}$

$\phi \longmapsto \mathbb{Z}$

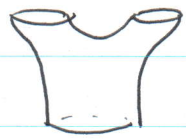
On morphisms

$\uparrow$

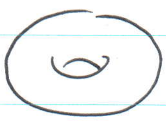
 $\mathbb{Z} \xrightarrow{i} A$ unit map

 $A \otimes A \xrightarrow{m} A$ multiplication

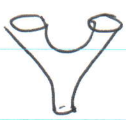
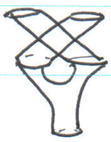
 $A \to \mathbb{Z}$ counit (trace)

 $A \to A \otimes A$ comultiplication

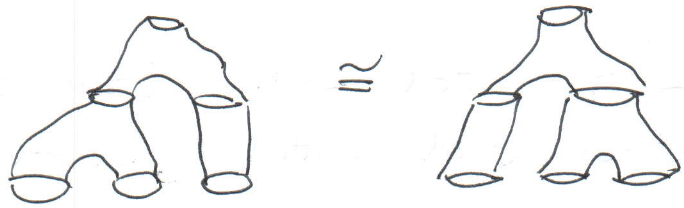
The above example

 $\mathbb{Z} \xrightarrow{2 = \dim A} \mathbb{Z}$

 $= 0$ (nilpotency)

N.B.  $\underset{\text{diffeo}}{=}$  commutative multiplication

京都大学大学院理学研究科数学教室

associativity 

$A$ is graded with $\deg(1) = -1$, $\deg(X) = 1$ — shifted from the usual degree

check that $\deg m = \deg \Delta = 1$
$\deg \varepsilon = \deg i = -1$

Observation $\deg = -\chi$

$\Rightarrow$ For an arb. 2-dim. cobordism $S$
$\deg S = -\chi(S)$

Our TQFT

$2\,\text{cob.} \xrightarrow{\mathcal{F}} \text{graded Abelian groups}$

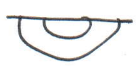
The rings $H^n$ is the sum of $\mathcal{F}$ applied to all gluings of crossingless matches of $2n$ dots

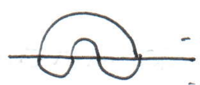
$B^n = \{\text{crossingless matches}\}$

Given $a \in B^n$  define $W(a) =$ 

reflection

If $a, b \in B^n$, then $W(b)a$ is a bunch of circles

$a =$  $b =$ 

$W(b)a =$ 

NB.

Orientation

$\circ$ ∞から交わる回数で定める.

Definition

$$H^n = \bigoplus_{a,b \in B^n} \mathcal{F}(w(b)a)\{n\}$$

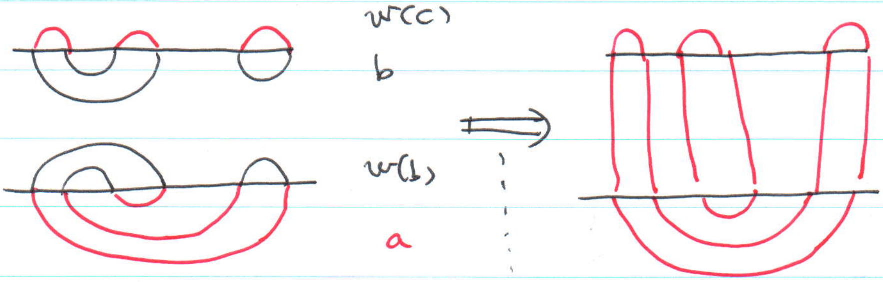
grading is shifted by $n$

multiplication $H^n \otimes H^n \longrightarrow H^n$

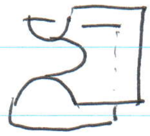
$$\mathcal{F}(w(d)c) \otimes \mathcal{F}(w(b)a) \longrightarrow 0 \quad \text{if } c \neq b$$

$$\mathcal{F}(w(c)b) \otimes \mathcal{F}(w(b)a) \xrightarrow{\mathcal{F}(S)} \mathcal{F}(w(c)a)$$

In $H^3$




Cobordism $S$

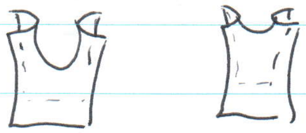


Note $\chi(S) = -n$

on $H^n$

$\Rightarrow$ mult. grading preserving

associativity $\leftarrow$



上下をずらす

by functoriality of $\mathcal{F}$

Identities $\quad 1_a := 1^{\otimes n} \in A^{\otimes n} \cong \mathcal{F}(w(a)a)$

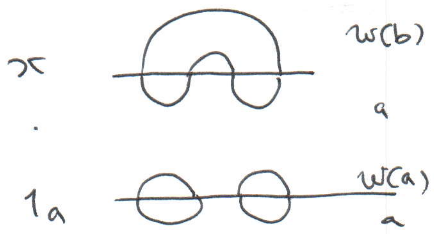
for some $a \in B^n$

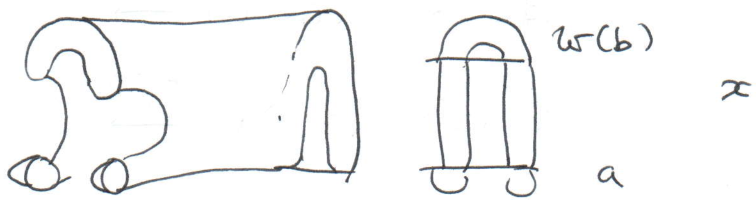
The collection of $1_a$ are orthogonal idempotent

$$1_a x = \begin{cases} x & \text{if } x \in \mathcal{F}(w(a)b) \\ 0 & \text{otherwise} \end{cases}$$

$$x 1_a = \begin{cases} x & \text{if } x \in \mathcal{F}(w(b)a \\ 0 & \end{cases}$$

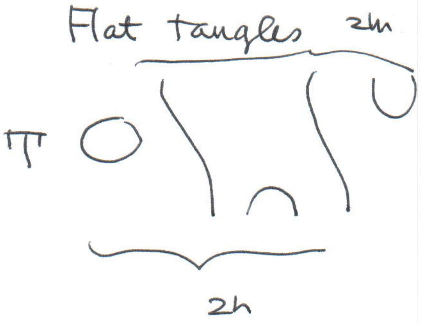
$$1_a 1_a = 1_a \qquad 1_a 1_b = 0 \quad a \neq b$$



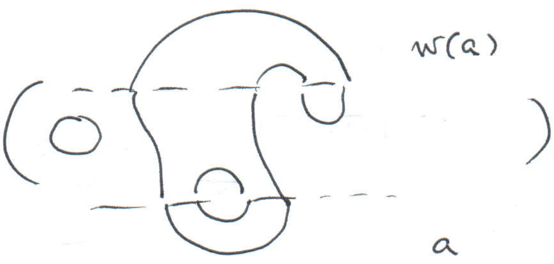

— . —

Flat tangles

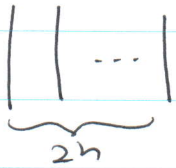



$$\overset{\mathcal{F}}{\longmapsto} \quad H^m \uparrow H^n \quad (H^n, H^m)\text{-bimodules } \mathcal{F}(T)$$

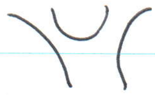
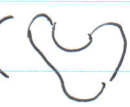
$$\mathcal{F}(T) = \bigoplus_{\substack{a \in B^n \\ b \in B^m}} \mathcal{F}(w(b)Ta)\{n\}$$

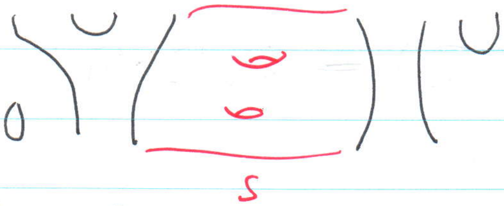
$\mathcal{F}($  $)$

**Example**

-  $\mathcal{F}(T) = H^n$

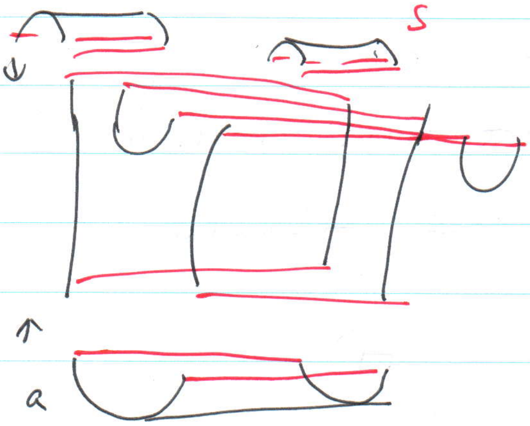


-  $\mathcal{F}(T) = \mathcal{F}($$)\{1\} \oplus \mathcal{F}($$)\{1\}$

flat tangle cobordism 



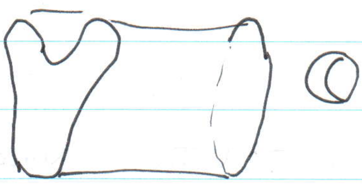
$\mapsto$ ? bimodule map $\mathcal{F}(T_1) \to \mathcal{F}(T_2)$





$a \times [0,1]$ $\quad a \in B^n$

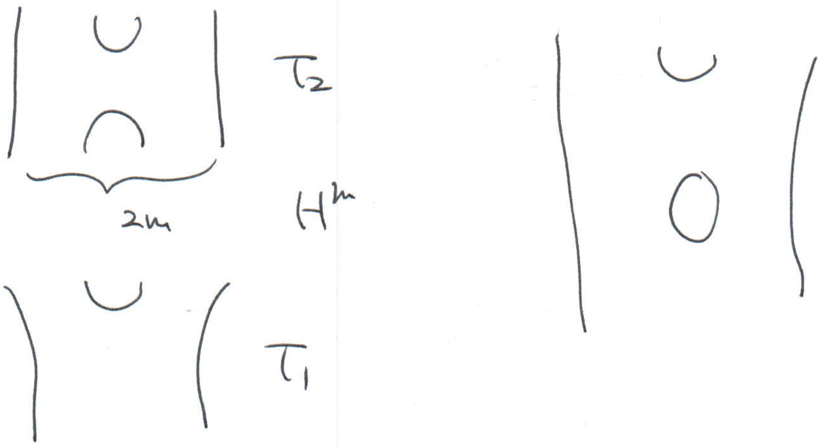
obtain a cobordism in $\mathbb{R}^3$ between closed 1-mfd



sum over $a, b$ and apply $\mathcal{F}$

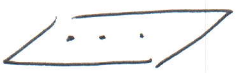
$$\mathcal{F}(T_1) \xrightarrow{\mathcal{F}(S)} \mathcal{F}(T_2)$$

a bimodule map




$$\mathcal{F}(T_2)\otimes_{H^m}\mathcal{F}(T_1) = \mathcal{F}(T_2\circ T_1)$$

Thus $\mathcal{F}$ is a 2-functor from
2-category of flat tangle cobordisms
$\longrightarrow$ 2-category of $H^n$-modules

obj  $\longmapsto$ $H^n$

mor  $\longmapsto$ $(H^n, H^m)$-bimodules

2-mor  $\longmapsto$ bimodule map

To add a fourth dimensions
go from modules $\to$ complexes of modules

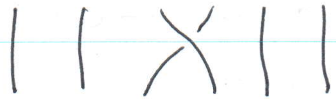
$\mathbb{R}^3$

$H^n$-modules $\longrightarrow$ cpx of $H^n$-modules

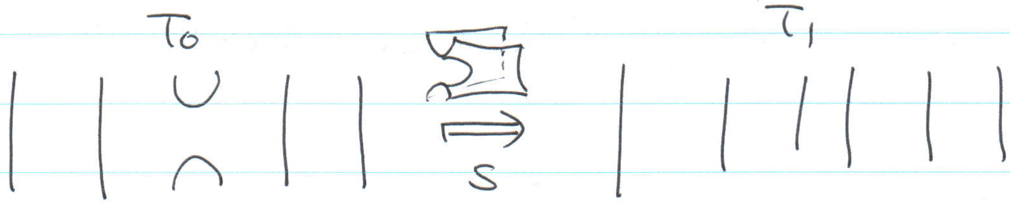
abelian categories $\longrightarrow$ triangulated categories

Any $(H^n, H^m)$-bimodule $N$ defines a functor

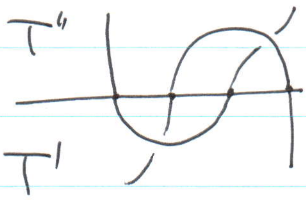
$$H^n\text{-modules} \xrightarrow[N\otimes_{H^n}]{} H^m\text{-modules}$$

$T_0$  $T$

we associate a complex of bimodules



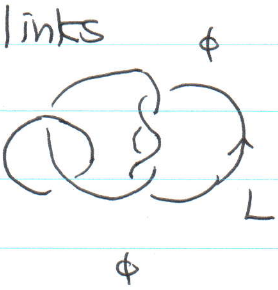

$$0 \longrightarrow \mathcal{F}(T_0) \xrightarrow{\mathcal{F}(S)} \mathcal{F}(T_1)\{-1\} \longrightarrow 0$$




$$\mathcal{F}(T) = \mathcal{F}(T'') \otimes_{H^m} \mathcal{F}(T')$$

<u>Thm</u>. $\mathcal{F}(T)$ is invariant under Reidemeister moves (up to homotopy)

links




$$\begin{array}{c} K(H^0) \\ \uparrow \mathcal{F}(L) \\ K(H^0) \\ \shortparallel \\ \mathbb{Z} \end{array}$$

a complex of graded abelian groups

Take cohomology of $\mathcal{T}(L)$

then the graded dimensions of each

cohomology group is an invariant of $L$

$$Kh(L) = \bigoplus_{i,j \in \mathbb{Z}} H^{i,j}(L) \quad \text{(doubly graded)}$$

homological grading

$$\text{Jones}(L) = \sum_{i,j} (-1)^i q^j \operatorname{rank} H^{i,j}(L)$$

(graded Euler characteristic)

<u>Rem</u>. This is stronger than Jones polynomial